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COMPUTING THE GENUS OF THE 2-AMALGAMATIONS OF GRAPHS

R. W. DECKER, H. H. GLOVER and J. P. HUNEKE

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The above authors [2] and S. Stahl [3] have shown that if a graph G is the 2-amalgamation of subgraphs G_1 and G_2 (namely if $G=G_1\cup G_2$ and $G_1\cap G_2=\{x, y\}$, two distinct points) then the orientable genus of G, $\gamma(G)$, is given by $\gamma(G)=\gamma(G_1)+\gamma(G_2)+\varepsilon$, where $\varepsilon=0, 1$ or -1. In this paper we sharpen that result by giving a means by which ε may be computed exactly. This result is then used to give two irreducible graphs for each orientable surface.

0. Introduction

Let G be a finite graph (loops and multiple edges allowed) and let $\gamma(G)$ denote its orientable genus. Let x and y denote two distinct points on G and let K_n denote the complete graph on n vertices. Define $G'=G\cup K_2$ with $G\cap K_2=\{x, y\}$ and define $G''=G\cup K_5$ with $G\cap K_5=\{x, y\}$, where x and y are also two points on the interiors of nonadjacent edges of K_5. Define

$$\mu(G)=\mu(G;x,y)=\begin{cases}0 & \text{if } \gamma(G)=\gamma(G')-1=\gamma(G'')-2\\ 1 & \text{if } \gamma(G)=\gamma(G')-1=\gamma(G'')-1\\ 2 & \text{if } \gamma(G)=\gamma(G')=\gamma(G'')-1\\ 3 & \text{if } \gamma(G)=\gamma(G')=\gamma(G'').\end{cases}$$

Suppose that G_1 and G_2 are subgraphs of a graph G such that $G=G_1\cup G_2$ and $G_1\cap G_2=\{x, y\}$. In such a case we say that G is the *2-amalgamation* of G_1 and G_2 and we write $G=G_1 \underset{\{x,y\}}{\cup} G_2$.

Define ε by the table below

	ε	$\mu(G_1)$: 0	1	2	3
$\mu(G_2)$	0	1	1	1	1
	1	1	1	1	0
	2	1	1	0	0
	3	1	0	0	-1

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We will prove:

Theorem 0.1. $\gamma(G_1 \bigcup_{\{x,y\}} G_2) = \gamma(G_1) + \gamma(G_2) + \varepsilon.$

Remark. For a real number r, let $\{r\}$ denote the integer hull of r, the least integer greater than or equal to r. The value of ε can be computed by the formula:

$$\varepsilon = \left\{(3 - \mu(G_1)\mu(G_2))/4\right\}.$$

Let S_g denote the orientable surface (closed 2-manifold) of genus g. Recall that a graph G is called *irreducible* for S_g if G minimally fails to embed in S_g, which is to say $G \not\subset S_g$ but $H \subset S_g$ for all proper subgraphs H of G. Let

$$nG = \bigcup_{\substack{\{x,y\} \\ i=1}}^{n} G$$

denote the 2-amalgamation of n copies of G at the two points $x, y \in G$. Using Theorem 0.1 we construct two irreducible graphs for each orientable surface in the following result.

Corollary 0.2. *Choose x and y to be adjacent vertices on either K_5 or $K_{3,3}$ and let e denote the (x, y) edge. Then $(2g+1)(K_5 - e)$ and $(2g+1)(K_{3,3} - e)$ are irreducible for S_g.*

Remark. Corollary 0.2 is a special case of the general conjecture that every irreducible graph for S_g is a topological union of $2g+1$ of the two Kuratowski graphs K_5 and $K_{3,3}$.

In section 1, respectively section 2, following we prove Theorem 0.1, respectively Corollary 0.2.

1. The proof of Theorem 0.1

Let D denote the closed unit disk in $\mathbf{R}^2$. Let $x_1, y_1, \ldots, x_m, y_m$ be $2m$ alternating ponts on the boundary of D and let $\sim$ denote the equivalence relation generated by $x_i \sim x_j$, $y_i \sim y_j$, $1 \leqq i, j \leqq m$. Call D_m, the quotient space $D/\sim$, a *2-identified disk*, and say that the *alternation number* of D_m is m. For any topological space X define the genus of X, $\gamma(X)$, exactly as for a graph.

Lemma 1.1. [2] *Let D_{n_i}, be 2-identified disks with alternation numbers n_i, $i = 1, \ldots, k$, then*

$$\gamma(\bigcup_{\{x,y\}} D_{n_i}) = \left\{\sum_{i=1}^{k} (n_i - 1)/2\right\}. \quad ∎$$

Corollary 1.2. *Let $\bigcup_{\{x,y\}} D_{n_i} \subset S$ be an embedding on a surface with $\gamma(S) = \left\{\sum_{i=1}^{k} (n_i - 1)/2\right\}$. Then there exist $\sum_{i=1}^{k} n_i$ complementary regions with two bounding 1-cells if the expression*

$\sum_{i=1}^{k}(n_i-1)$ *is even. Otherwise there exist* $(\sum_{i=1}^{k} n_i)-1$ *complementary regions, one with four bounding* 1*-cells and the rest with two bounding* 1*-cells.* ∎

An embedding $X\subset S$ of a space X in a surface S is called a *2-cell embedding* if each component of $S-X$ is an open 2-cell (is homeomorphic to $\mathbf{R}^2$). Observe that if $\bigcup_{\{x,y\}} D_{n_i}\subset S$ is a 2-cell embedding then the closure of each component of $S-\bigcup_{\{x,y\}} D_{n_i}$ is also a 2-identified disk.

Proposition 1.3. [2] *Let S and S' be surfaces and for each $i=1, \ldots, r$, let both D_{n_i} and D'_{n_i} be 2-identified disks with the same alternation number. Suppose that $\bigcup_{\{x,y\}} D_{n_i}\subset S$ and $\bigcup_{\{x,y\}} D'_{n_i}\subset S'$ are 2-cell embeddings with complementary regions C_{m_j}, $j=1, \ldots, t$ and C'_{m_k}, $k=1, \ldots, t'$, respectively (which are 2-identified disks by the remark above). Then there exists a surface T such that $\bigcup_{\{x,y\}} C_m \bigcup_{\{x,y\}} \bigcup_{\{x,y\}} C'_{m_k}\subset T$ with*

$\gamma(T)=\gamma(S)+\gamma(S')-\sum_{i=1}^{r}(n_i-1).$ ∎

Observe that if $\{x, y\}\in G\subset S$ is a 2-cell embedding of a graph G it is possible to define the alternation number with respect to $\{x, y\}$ of each of the components of $S-G$. A component C of $S-G$ is said to have *alternation number* m if $D_m\subset C\cup\{x, y\}$ but $D_{m+1}\not\subset C\cup\{x, y\}$. Thus for example if in tracing the boundary of C one finds exactly the cycle of points $xyxxyyx$, it is then clear that the alternation number of C with respect to $\{x, y\}$ is 2.

In our application to Corollary 0.2 we will use the following result.

Lemma 1.4. [2] *Let G be a graph, S a surface and $G\subset S$ a 2-cell embedding. Choose a pair of distinct points $x, y\in G$ and let n_i, $i=1, \ldots, r$, denote the alternation numbers with respect to $\{x, y\}$ of the components of $S-G$. Then*

$$\{(\sum_{i=1}^{r}(n_i-1)-1)/2\}\leqq\gamma(S)-\gamma(G). \quad ∎$$

Corollary 1.5. *If $G\subset S$ is a genus embedding then there can be at most one complementary region with alternation number two and none higher. Furthermore, if $G\subset S$ is any 2-cell embedding then the alternation number of any complementary region is at most $2+2(\gamma(S)-\gamma(G))$.*

Recall now the definition of $\mu(G)$ from the introduction. The following result shows the connection between $\mu(G)$ and the complementary alternation numbers for a graph embedding.

Proposition 1.6. *Let $\overline{m}=(m_1, \ldots, m_s)$ be an s-tuple of integers $m_i\geqq 1$. For $i=1, \ldots, s$ and a graph G, let $a(G, \overline{m})$ be the least integer such that if S is a surface with $\gamma(S)=\gamma(G)+a(G, \overline{m})$ then there exists an embedding $G\subset S$ such that there are disjoint 2-identified disks, $Q_1, \ldots, Q_s$ in $(S-G)\cup\{x, y\}$ which have alternation numbers*

$m_1, \ldots, m_s$ with respect to fixed vertices $x, y \in VG$. Then

$$a(G, \bar{m}) = \left\{(2-\mu(G)+\sum_{i=1}^{s}(m_i-1))/2\right\}.$$

Proof. By Lemma 1.1 we may embed $\bigcup_{i=1}^{s} Q_i \subset S'$ where $Q_i \cap Q_j = \{x, y\}$, $i \neq j$, $i, j = 1, \ldots, s$ and where $\gamma(S') = \left\{\sum_{i=1}^{s}(m_i-1)/2\right\}$. The surface S' will be fixed throughout the proof. By Corollary 1.2 all components of $S' - \bigcup_{i=1}^{s} Q_i$ will have alternation number 1 if $\sum_{i=1}^{s}(m_i-1)$ is even and exactly 1 complementary region will have alternation number 2 if $\sum_{i=1}^{s}(m_i-1)$ is odd. From the definition of $\mu(G)$ for the special case where $\bar{m}=(1)$ it is clear that $a(G, 1) = \{(2-\mu(G))/2\}$ and in case $\bar{m}=(2)$ it is clear that $a(G, 2) = \{(3-\mu(G))/2\}$. To establish an upper bound for $a(G, \bar{m})$ we consider 2 cases, depending upon the parity of $\sum_{i=1}^{s}(m_i-1)$.

Case 1. $\sum_{i=1}^{s}(m_i-1)$ *is even.*

Embed $G \subset S$ such that $\gamma(S) = \gamma(G) + a(G, 1) = \gamma(G) + \{(2-\mu(G))/2\}$ and such that $(S-G) \cup \{x, y\}$ contains a 2-identified disk Q with alternation number 1. Assuming $m_1 \geqq 1$ there exists a 2-cell $Q' \subset Q_1 \subset S'$ with alternation number 1. Using Proposition 1.3, attach $S-Q$ and $S'-Q'$ along the boundary of Q (the ordinary connected sum in this case) to form a surface T with

$$\gamma(T) = \gamma(S)+\gamma(S') = \gamma(G)+\{(2-\mu(G))/2\}+\left\{\sum_{i=1}^{s}(m_i-1)/2\right\} =$$

$$= \gamma(G)+\left\{(2-\mu(G)+\sum_{i=1}^{s}(m_i-1))/2\right\}.$$

By construction, $G \subset T$ so $a(G, \bar{m}) \leqq \left\{(2-\mu(G)+\sum_{i=1}^{s}(m_i-1))/2\right\}$.

Case 2. $\sum_{i=1}^{s}(m_i-1)$ *is odd.*

Embed $G \subset S$ so that $\gamma(S) = \gamma(G) + a(G, 2) = \gamma(G) + \{(3-\mu(G))/2\}$ and so that $(S-G) \cup \{x, y\}$ contains a 2-cell Q with alternation number 2. As in Case 1 use Proposition 1.3 to construct a surface T with

$$\gamma(T) = \gamma(S)+\gamma(S')-1 = \gamma(g)+\{(3-\mu(G))/2\}+\left\{\sum_{i=1}^{s}(m_i-1)/2\right\}-1 =$$

$$= \gamma(G)+\left\{(2-\mu(G)+\sum_{i=1}^{s}(m_i-1))/2\right\},$$

so that again

$$a(G, \overline{m}) \leqq \{(2-\mu(G)+\sum_{i=1} (m_i-1))/2\}. \blacksquare$$

We next show $a(G, m) \geqq \{(2-\mu(G)+\sum_{i=1}^{s} (m_i-1))/2\}$. Suppose $a(G, \overline{m}) \leqq \leqq \{(2-\mu(G)+\sum_{i=1}^{s} (m_i-1))/2\}-1$. Embed G on a surface S with complementary alternations $m_1, \ldots, m_s$ and with $\gamma(S)-\gamma(G)=a(G, \overline{m})$. Again, using Proposition 1.3, attach S and S' along the 2-cells $Q_1, \ldots, Q_s$, forming a surface T with

$$\begin{aligned}
\gamma(T) &= \gamma(S)+\gamma(S')-\sum_{i=1}^{s} (m_i-1) \\
&= \gamma(S)+\{\sum_{i=1} (m_i-1)/2\}-\sum_{i=1} (m_i-1) \\
&= \gamma(S)-\{((\sum_{i=1} (m_i-1))-1)/2\} \\
&\leqq \gamma(G)+\{(2-\mu(G)+\sum_{i=1}^{s} (m_i-1))/2\}-1-\{(\sum_{i=1}^{s} (m_i-1)-1)/2\} \\
&= \gamma(G)+\begin{cases} \{(1-\mu(G))/2\} & \text{if } \sum_{i=1}^{s} (m_i-1) \text{ is odd} \\ \{-\mu(G)/2\} & \text{if } \sum_{i=1}^{s} (m_i-1) \text{ is even.} \end{cases}
\end{aligned}$$

By construction, $G \subset S - \bigcup_{i=1}^{s} Q_i \subset T$ so $\gamma(G) \leqq \gamma(T)$. Again, we have 2 cases, depending upon the parity of $\sum_{i=1}^{s} (m_i-1)$.

Case 1'. $\sum_{i=1}^{s} (m_i-1)$ *is even.*

$S' - \bigcup_{i=1}^{s} Q_i$ contains a complementary 1-alternation and $\gamma(T) \leqq \gamma(G)$ in case $\mu(G)=0$ or 1, which together contradict $a(G, 1)=1$. If $\mu(G)=2$ or 3 we have $\gamma(G) \leqq \gamma(T) \leqq \gamma(G)-1$, a contradiction.

Case 2'. $\sum_{i=1}^{s} (m_i-1)$ *is odd.*

In this case $S' - \bigcup_{i=1}^{s} Q_i$ contains a 2-alternation which was not eliminated by attaching S to S' and Case 2 above implies $\gamma(T) \leqq \gamma(G)+1$ (resp. $\leqq \gamma(G)$, $\leqq \gamma(G)$) when $\mu(G)=0$ (resp. 1, 2). However, in each of these three possibilities we have a contradiction to $a(G,2)=2$ (resp. 1, 1). In case $\mu(G)=3$ we have, by the remark above, $\gamma(G)\leqq\gamma(T)\leqq\gamma(G)-1$, clearly a contradiction. ▮

Corollary 1.7. *Let $G\subset S$ be a 2-cell embedding with complementary components having alternation numbers $m_1, \ldots, m_s\geqq 1$. Then*

$$\left\{\left(2-\mu(G)+\sum_{i=1}^{s}(m_i-1)\right)/2\right\}\leqq\gamma(S)-\gamma(G). \quad ▮$$

It should be noted that if $\mu(G)=0$ or 1, Corollary 1.7 implies that there can be no genus embeddings with complementary alternations $\geqq 1$.

Theorem 1.8. *Let G_1 and G_2 be two graphs such that $G_1\cap G_2=\{x,y\}$. Then*

$$\gamma\left(G_1\bigcup_{\{x,y\}}G_2\right)\leqq\gamma(G_1)+\gamma(G_2)+\left\{\left(3-\mu(G_1)\mu(G_2)\right)/4\right\}.$$

Proof. Note $\gamma(G_1)+\gamma(G_2)+1$ is an upper bound for $\gamma\left(G_1\bigcup_{\{x,y\}}G_2\right)$. This can be seen easily by taking genus embeddings of G_1 and G_2 on two surfaces and then attaching the surfaces with two tubes. Thus only the cases $(\mu(G_1), \mu(G_2))=(1,3), (2,2), (2,3)$ or $(3,3)$ need to be considered.

Case 1. $(\mu(G_1), \mu(G_2))=(1,3), (2,3)$ or $(3,3)$.

Let $m\geqq 1$ be an arbitrary even integer. By Proposition 1.6 we may embed $G_1\subset S$ and $G_2\subset S'$ such that $\gamma(S)=\gamma(G_1)+\left\{\left(1+m-\mu(G_1)\right)/2\right\}$ and $\gamma(S')=\gamma(G_2)+\left\{\left(1+m-\mu(G_2)\right)/2\right\}$ and with $S-G_1$ and $S'-G_2$ containing 2-cells Q and Q' with alternation number m. By Proposition 1.3 we may attach S and S' along the m-alternations to yield a surface T with

$$\begin{aligned}\gamma(T) &= \gamma(S)+\gamma(S')-(m-1)\\ &= \gamma(G_1)+\left\{\left(1+m-\mu(G_1)\right)/2\right\}+\gamma(G_2)+\left\{\left(1+m-\mu(G_2)\right)/2\right\}-(m-1)\\ &= \gamma(G_1)+\left\{\left(1-\mu(G_1)\right)/2\right\}+m/2+\gamma(G_2)+\left\{\left(1-\mu(G_2)\right)/2\right\}+m/2-(m-1),\end{aligned}$$

since m is even. Thus we have

$$\gamma(T)=\gamma(G_1)+\gamma(G_2)+\left\{\left(1-\mu(G_1)\right)/2\right\}+\left\{\left(1-\mu(G_2)\right)/2\right\}+1.$$

Substituting $(\mu(G_1), \mu(G_2))=(1,3), (2,3)$ or $(3,3)$ respectively, we see that $\gamma(T)=\gamma(G_1)+\gamma(G_2)$, $\gamma(G_1)+\gamma(G_2)$ or $\gamma(G_1)+\gamma(G_2)-1$ respectively as required.

Case 2. $\mu(G_1)=\mu(G_2)=2$.

A repeat of the argument of Case 1 with m an arbitrary odd integer yields the results in this case. ▮

We now establish lower bounds for the genus of a 2-amalgamation of graphs, thus proving Theorem 0.1. To this end we prove the following result.

Theorem 1.9. *Let G_1 and G_2 be two graphs such that $G_1\cap G_2=\{x,y\}$, then*

$$\gamma\left(G_1\bigcup_{\{x,y\}}G_2\right)\geqq\gamma(G_1)+\gamma(G_2)+\left\{\left(3-\mu(G_1)\mu(G_2)\right)/4\right\}.$$

Proof. Let $G_1 \bigcup_{\{x,y\}} G_2 \subset S$ be a genus embedding. This induces an embedding $G_2 \subset S$ for which G_1 lies in components $C_1, \ldots, C_s$ of $S - G_2$. Suppose that among these components there are h handles. Cap those handles, producing a new embedding $G_2 \subset S'$ with $\gamma(S') = \gamma(S) - h$. In this embedding, the components $C_1, \ldots, C_s$ have been replaced by 2-cells $C_1', \ldots, C_t'$. Let the alternation numbers of $C_1', \ldots, C_t'$ be denoted by $m_1, \ldots, m_t$. There are two cases to be considered, depending upon whether or not $\sum_{i=1}^{t} m_i = 0$.

Suppose first that $m_i = 0$ for $i = 1, \ldots, t$. Then we could embed $\bigcup_{i=1}^{t} C_i' \subset S_0$, the surface of genus 0 and replace the handles, thus $\bigcup_{i=1}^{s} C_i \subset S_h$ and since $G_i \subset \bigcup_{i=1}^{s} C_i \subset S_h$ we conclude $\gamma(G_1) \leqq h$. Observe that in placing $\bigcup_{i=1}^{t} C_i' \subset S_0$ we could also embed an (x, y) edge in $(S_0 - \bigcup_{i=1}^{t} C_i') \cup \{x, y\}$, hence also in $(S_h - \bigcup_{i=1}^{s} C_i) \cup \{x, y\}$ so, in the notation from the introduction, $\gamma(G_1') \leqq h$. Since, by the definition of μ, $\gamma(G_1') = \gamma(G_1) + \{(2 - \mu(G_1))/2\}$ we have $\gamma(G_1) + \{(2 - \mu(G_1))/2\} = \gamma(G_1') \leqq h$ so

$$\gamma(G_1) \leqq h - \{(2 - \mu(G_1))/2\}. \tag{1}$$

Now consider the embedding $G_2 \subset S'$. Let $n_1, \ldots, n_r$ represent all nonzero alternations of components of $S' - G_2$, then by Corollary 1.7

$$\begin{aligned}\{(2 - \mu(G_2))/2\} &\leqq \{(2 - \mu(G_2) + \sum_{i=1}^{r} (n_i - 1))/2\} \\ &\leqq \gamma(S') - \gamma(G_2) \\ &= \gamma(G_1 \textstyle\bigcup_{\{x,y\}} G_2) - h - \gamma(G_2)\end{aligned}$$

so

$$\gamma(G_2) \leqq \gamma(G_1 \textstyle\bigcup_{\{x,y\}} G_2) - h - \{(2 - \mu(G_2))/2\}. \tag{2}$$

Adding (1) and (2) we obtain, in this case

$$\gamma(G_1) + \gamma(G_2) \leqq \gamma(G_1 \textstyle\bigcup_{\{x,y\}} G_2) - \Phi$$

where

$$\Phi = \{(2 - \mu(G_1))/2\} + \{(2 - \mu(G_2))/2\}. \tag{3}$$

By enumerating the possible values for $(\mu(G_1), \mu(G_2))$ we see in each case that $\Phi \geqq \{(3 - \mu(G_1)\mu(G_2))/4\}$, so that

$$\gamma(G_1 \textstyle\bigcup_{\{x,y\}} G_2) \geqq \gamma(G_1) + \gamma(G_2) + \{(3 - \mu(G_1)\mu(G_2))/4\},$$

if $\sum_{i=1}^{t} m_i = 0$.

Suppose now that for this embedding that there exists an $m_i > 0$. Without loss of generality we may consider $m_i > 0$ for $i = 1, \dots, u$ and $m_i = 0$ for $i = u+1, \dots, t$. By Lemma 1.1 we may embed $\bigcup\limits_{i=1}^{u} C_i'$ on a surface T' with $\gamma(T') = \left\{\sum\limits_{i=1}^{u} (m_i - 1)/2\right\}$. In this embedding, Corollary 1.2 yields the result that there is a 2-cell $R \subset T' - \bigcup\limits_{i=1}^{u} C_i'$ with alternation number 1 if $\sum\limits_{i=1}^{u} (m_i - 1)$ is even and with alternation number 2 otherwise. In case $\sum\limits_{i=1}^{u} (m_i - 1)$ is even, we may place an (x, y) edge in R and in case $\sum\limits_{i=1}^{u} (m_i - 1)$ is odd, we could place K, our special copy of K_5, in R. In either case, it is clear that we could also embed $\bigcup\limits_{i=u+1}^{t} C_i' \subset R - e$ (resp. $R - K$). If we now replace the h handles, we see that $G_i' \subset \bigcup\limits_{i=1}^{t} C_i \subset T$ in case $\sum\limits_{i=1}^{u} (m_i - 1)$ is even and $G_1'' \subset \bigcup\limits_{i=1}^{t} C_i \subset T$ if $\sum\limits_{i=1}^{u} (m_i - 1)$ is odd. Thus we conclude $\gamma(G_1')$ (resp. $\gamma(G_1'')) \leqq \left\{\sum\limits_{i=1}^{u} (m_i - 1)/2\right\} + h$ if $\sum\limits_{i=1}^{u} (m_i - 1)$ is even (resp. odd).

As before, the definition of μ implies that $\gamma(G_1') = \gamma(G_1) + \{(2 - \mu(G_1))/2\}$ and $\gamma(G_1'') = \gamma(G_1) + \{(3 - \mu(G_1))/2\}$. Combining these with the results of the preceding paragraph yields

$$(4) \qquad \gamma(G_1) \leqq \left\{\left(\sum_{i=1}^{u} (m_i - 1)\right)/2\right\} + h - \begin{cases} \{(2 - \mu(G_1))/2\} & \text{if } \sum\limits_{i=1}^{u} (m_i - 1) \text{ even} \\[2ex] \{(3 - \mu(G_1))/2\} & \text{if } \sum\limits_{i=1}^{u} (m_i - 1) \text{ odd.} \end{cases}$$

Now consider the embedding $G_2 \subset S'$. Let $n_1, \dots, n_r$ denote all nonzero alternations of components of $S' - H$, then by Corollary 1.7

$$\begin{aligned} \left\{\left(2 - \mu(G_2) + \sum_{i=1}^{u} (m_i - 1)\right)/2\right\} &\leqq \left\{\left(2 - \mu(G_2) + \sum_{i=1}^{r} (n_i - 1)\right)/2\right\} \leqq \\ &\leqq \gamma(S') - \gamma(G_2) \\ &= \gamma\left(G_1 \bigcup_{\{x,y\}} G_2\right) - h - \gamma(G_1) \end{aligned}$$

so

$$(5) \qquad \gamma(G_2) \leqq \gamma\left(G_1 \bigcup_{\{x,y\}} G_2\right) - h - \left\{\left(2 - \mu(G_2) + \sum_{i=1} (m_i - 1)\right)/2\right\}.$$

Adding (4) and (5) and simplifying, we have

$$\gamma(G_1 \underset{\{x,y\}}{\cup} G_2) \geqq \gamma(G_1)+\gamma(G_2)+\begin{cases} \Phi & \text{if } \sum_{i=1}^{u}(m_i-1) \text{ even} \\ \Psi & \text{if } \sum_{i=1}^{u}(m_i-1) \text{ odd,} \end{cases}$$

where Φ is as in (3) and

$$\Psi = \{(3-\mu(G_1))/2\}+\{(1-\mu(G_2))/2\}.$$

Again, enumerating the possible values for $(\mu(G_1), \mu(G_2))$ we see that in each case Φ, $\Psi \geqq \{(3-\mu(G_1)\mu(G_2))/4\}$ thus completing the proof of Theorem 1.9. ∎

Proof of Theorem 0.1. By Theorem 1.8 $\gamma(G_1 \underset{\{x,y\}}{\cup} G_2) \leqq \gamma(G_1)+\gamma(G_2)+\{(3-\mu(G_1)\times \times\mu(G_2))/4\}$. By Theorem 1.9 the reverse inequality holds. As a result we have now the desired equality using the Remark following Theorem 0.1. ∎

We now consider two questions posed in Alpert's Thesis [1] in which he considers the amalgamation of two graphs G and H denoted $GV_{K_2}H$ by identifying an edge in each graph. It is clear that the genus of the resulting graph would be the same as the genus of the graph obtained by identifying only endpoints of the edges in G and H. We then have

Corollary 1.10. (cf. [1], p. 37) $\gamma(GV_{K_2}H) \geqq \gamma(G)+\gamma(H)-1$.

Corollary 1.11. ([1], p. 20) *If* $\gamma(GV_{K_2}H)=\gamma(G)+\gamma(H)-1$ *then there exist genus embeddings* $G \subset S$, $H \subset T$ *such that* $S-G$ *and* $T-H$ *both contain complementary 2-alternations.*

Proof. By Theorem 0.1 $\varepsilon=-1$ only if $\mu(G)=\mu(H)=3$, then Proposition 1.6 implies that there are genus embeddings of G and H of the desired type. ∎

2. The proof of Corollary 0.1.

We recall some notation from the introduction. Suppose that $K=K_5\cup\{x, y\}$, where x, y are on the interior of non-adjacent edges of K_5. For a graph G with $x, y \in VG$, $G''=G\cup K$, $G\cap K=\{x, y\}$ and $G'=G\cup e$ where e is an edge with endpoints x, y. Finally, recall that if G is a graph with vertices x, y then nG is obtained by identifying n homeomorphic copies of G at a pair of vertices x, y.

Lemma 2.1. *If K is as above,* $\mu(2K)=2$ *and* $\mu(K')=3$.

Proof. Figure 1 below shows that $\gamma(K\cup e\cup K)=\gamma(K\cup e)=1$ so $\mu(K\cup e)=3$. Also the figure shows that $\gamma(2K\cup e)=\gamma(2K)$ so $\mu(2K)\geqq 2$. If $\mu(2K)=3$ then $\gamma(2K\cup K)= =\gamma(2K)=1=\gamma(K)$ so if we embed $3K\subset T$, the torus, we may remove two copies of K and still be left with a genus embedding of the one remaining copy of K. Since the two copies of K could ony have been embedded in two 2-cells of $T-K$, each with alternation number at least 2 or one 2-cell with alternation number at least 3 we have a contradiction to Corollary 1.5. ∎

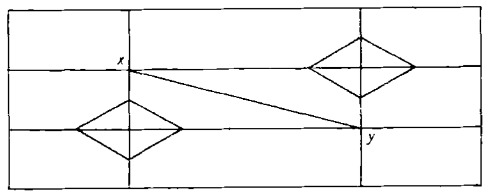

Fig. 1

The next lemma enables us to establish a connection between $\mu(G)$, $\mu(G')$, and $\mu(G'')$.

Lemma 2.2. *If $\mu(G)$ is even, then $\mu(G')=2$ and $\mu(G'')=3$. If $\mu(G)$ is odd, then $\mu(G')=3$ and $\mu(G'')=2$.*

Proof. First note that for any graph G, adding an (x, y) edge to G' or G'' will not change the genus of either graph so $\mu(G')$, $\mu(G'')\geqq 2$. To find when $\mu(G')$ or $\mu(G'')>2$ we consider $\gamma(G'\cup K)-\gamma(G')$ and $\gamma(G''\cup K)-\gamma(G'')$ where K is as above. By Theorem 0.1

$$\gamma(G\cup K) = \gamma(G\cup e\cup K) = \gamma(G)+\gamma(K\cup e)+\{(3-\mu(G)\mu(K\cup e))/4\} =$$
$$= \gamma(G)+\gamma(K')+\{(3-\mu(G)\mu(K'))/4\}.$$

It is easily seen from this and the definition of μ that

$$\gamma(G'\cup K)-\gamma(G') = \begin{cases} 1 & \text{if } \mu(G) = 0, 2 \\ 0 & \text{if } \mu(G) = 1, 3 \end{cases}$$

so, since $\mu(G')\geqq 2$ we have $\mu(G')=2$ if $\mu(G)=0,2$ and $\mu(G')=3$ if $\mu(G)=1, 3$. We next consider $\mu(G'')$. Again by Theorem 0.1 we have $\gamma(G''\cup K)=\gamma(G)+\gamma(2K)+$ $+\{(3-\mu(G)\mu(2K))/4\}$ so

$$\gamma(G''\cup K)-\gamma(G'') = \begin{cases} 0 & \text{if } \mu(G) = 0, 2 \\ 1 & \text{if } \mu(G) = 1, 3 \end{cases}$$

so as above $\mu(G'')=3$ if $\mu(G)=0, 2$ and $\mu(G'')=2$ if $\mu(G)=1, 3$. ∎

Theorem 2.3. *For arbitrary graphs G and H with $G\cap H=\{x, y\}$, $\mu(G\cup H)$ may be computed by the following table.*

Table 1

$\mu(G\cup H)$		$\mu(G)$			
		0	1	2	3
$\mu(H)$	0	0	1	2	3
	1	1	2	3	2
	2	2	3	2	3
	3	3	2	3	2

Proof. As in Lemma 2.2 we consider $\gamma(G\cup H\cup e)-\gamma(G\cup H)$ and $\gamma(G\cup H\cup K)-\gamma(G\cup H)$, where K is as above. By Theorem 0.1

$$\gamma(G\cup H\cup e)-\gamma(G\cup H)=\gamma(H\cup e)-\gamma(H)+\{(3-\mu(G)\mu(H\cup e))/4\}-$$
$$-\{(3-\mu(G)\mu(H))/4\}.$$

Denote $\gamma(G\cup H\cup e)-\gamma(G\cup H)$ by Φ. Similarly, let

$$\Psi=\gamma(G\cup H\cup K)-\gamma(G\cup H)=\gamma(H\cup K)-\gamma(H)+$$
$$+\{(3-\mu(G)\mu(H\cup K))/4\}-\{(3-\mu(G)\mu(H))/4\}.$$

For the 10 possible cases we tabulate Φ and Ψ, using Lemma 2.2.

Table 2

$\mu(G)$	$\mu(H)$	Φ	Ψ
0	0	1	2
0	1	1	2
0	2	0	1
0	3	0	0
1	1	0	1
1	2	0	0
1	3	0	1
2	2	0	1
2	3	0	0
3	3	0	1

Comparing Table 2 with Table 1 and using the definition of μ establishes the result. ∎

Corollary 2.4. *Let G be an arbitrary graph, then*
(i) *if $\mu(G)=0$, then $\gamma(nG)=n\gamma(G)+(n-1)$ and $\mu(nG)=0$,*
(ii) *if $\mu(G)=1$, then $\gamma(nG)=n\gamma(G)+\{n/2\}$ if $n>1$ and $\mu(nG)=3$ if $n>1$ is odd and $\mu(nG)=2$ if n is even,*
(iii) *if $\mu(G)=2$, then $\gamma(nG)=n\gamma(G)$ and $\mu(nG)=2$,*
(iv) *if $\mu(G)=3$, then $\gamma(nG)=n\gamma(G)-\{(n-1)/2\}$ and $\mu(nG)=3$ if n is odd and $\mu(nG)=2$ if n is even.*

Proof. The proof in each case is a simple induction argument using Theorems 0.1 and 2.3. ∎

Proof of Corollary. 0.3. Note first that $\gamma(K_5-e)=0$ and $\mu(K_5-e)=1$. Similarly $\gamma(K_{3,3}-e)=0$ and $\mu(K_{3,3}-e)=1$. Thus, by Corollary 2.4 $\gamma((2g+1)(K_5-e))=\gamma((2g+1)(K_{3,3}-e))=g+1$ and hence $(2g+1)(K_5-e)$ and $(2g+1)(K_{3,3}-e)$ will not embed on the surface S_g of genus g. Let H be any proper subgraph of $(2g+1)(K_5-e)$ or $(2g+1)(K_{3,3}-e)$. We claim that $H\subset S_g$. Let e^* be an edge of $(2g+1)(K_5-e)$ or $(2g+1)(K_{3,3}-e)$ and let K^* denote a copy of K_5-e or $K_{3,3}-e$ with e^* removed. We will show that $\gamma((2g+1)(K_5-e)-e^*)=\gamma(2g+1)(K_{3,3}{}^*-e)-e^*=g$. We may write $(2g+1)(K_5-e)-e^*=2g(K_5-e)\cup K^*$ and $(2g+1)(K_{3,3}-e)-e^*=2g(K_{3,3}-e)\cup K^*$ and note that $\gamma(K^*)=0$ and $\mu(K^*)=2$. Now apply

Theorem 0.1 and Corollary 2.4 to obtain

$$\begin{aligned}\gamma\big((2g+1)(K_5-e)-e^*\big) &= \gamma\big(2g(K_5-e)\cup K^*\big)\\ &= \gamma\big(2g(K_5-e)\big)+\gamma(K^*)+\big\{(3-\mu(2g(K_5-e))\mu(K^*))/4\big\}\\ &= g+0+\big\{(3-2\cdot 2)/4\big\}\\ &= g.\end{aligned}$$

A similar result holds for $\gamma\big((2g+1)(K_{3,3}-e)-e^*\big)$. ∎

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R. W. Decker

Hamilton College
Clinton, N.Y. 13 323
U.S.A.

H. H. Glover, J. P. Huneke

The Ohio State University
Columbus, Ohio 43 210
U.S.A.